FIAN/TD/7-06

hep-th/0605249

Axiomatic formulations of nonlocal and noncommutative field theories

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Abstract

We analyze functional analytic aspects of axiomatic formulations of nonlocal and noncommutative quantum field theories. In particular, we completely clarify the relation between the asymptotic commutativity condition, which ensures the CPT symmetry and the standard spin-statistics relation for nonlocal fields, and the regularity properties of the retarded Green's functions in momentum space that are required for constructing a scattering theory and deriving reduction formulas. This result is based on a relevant Paley-Wiener-Schwartz-type theorem for analytic functionals. We also discuss the possibility of using analytic test functions to extend the Wightman axioms to noncommutative field theory, where the causal structure with the light cone is replaced by that with the light wedge. We explain some essential peculiarities of deriving the CPT and spin-statistics theorems in this enlarged framework.

Keywords: nonlocal quantum fields, causality, noncommutative field theory, Wightman functions, analytic functionals, Paley-Wiener-Schwartz theorem

1. Introduction

This paper is concerned with some questions that arise in extending the Wightman axioms to nonlocal and noncommutative quantum field theories (QFTs) and are related to using enlarged classes of generalized functions suggested for this purpose instead of the tempered distributions used in the conventional formalism [1]-[3]. The generalized functions defined on the space S^0 of analytic test functions were perhaps proposed most

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often. This space is simply the Fourier transform of the Schwartz space $\mathcal{D}$ of infinitely differentiable functions of compact support and can serve as a functional domain for fields with an arbitrary high-energy behavior. The first question discussed below has a long history. In [4], Steinmann proposed replacing the local commutativity axiom, which loses its meaning for fields defined on S^0, by some regularity conditions on the retarded Green's functions in momentum space that ensure the existence of an S-matrix and provide a basis for developing a scattering theory. More recently, a generalization of microcausality in terms of coordinate space was found. This generalization, called asymptotic commutativity, ensures the standard spin-statistics relation and the CPT invariance of nonlocal QFT (see [5] and the references therein). The asymptotic commutativity is stated as the continuity property of the commutators of observable fields with respect to the topology of a space related to S^0 and associated with the closed light cone. In [6], a Paley-Wiener-Schwartz-type theorem was proved for the generalized functions defined on S^0, which allows clarifying the relation between asymptotic commutativity and Steinmann's conditions. But doing this, as shown below, requires further extending the theorem to multilinear forms.

The same mathematical technique proves useful for analyzing recent axiomatic formulations of noncommutative field theory. As noted in [7], such an axiomatization is somewhat premature because of the shortage of well-studied instructive examples. Actually, there is no general agreement regarding the situation with the Poincaré symmetry, causality and unitarity in noncommutative field theory, and this name is often used for theories with quite different physical contents. Here we discuss attempts to extend the Wightman axioms to field theories obtained from the Seiberg-Witten limit [8] of string theories; a feature of these field theories is the replacement of the causal structure with the light cone by that with the light wedge. Such a modification of the microcausality condition was proposed, for example, in [9] and was examined in more detail in [10], [11]. An analysis of this formulation is also interesting from the methodical standpoint. It shows that there is no need to use the whole Bargmann-Hall-Wightman theorem to derive the spin-statistics relation and the CPT invariance because it suffices to use its simplest version for the two-dimensional space-time, where the complex Lorentz group coincides with the group $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ of complex numbers. It was emphasized in [9], [11] that the framework of tempered distributions is hardly suitable for the theory developed by the authors in view of the singularities related to UV/IR mixing and because of the exponential growth of the momentum-space correlators along the noncommutative directions. In [12], generalized functions of the class S'^0 were proposed instead. In the second part of the present paper, we explain some subtleties in proving the CPT and spin-statistics theorems in such an enlarged framework and show that the most straightforward derivation of these theorems is by extending the uniqueness theorem obtained in [13] for the distributions supported in a properly convex cone to the case of a wedge-shaped support.

Section 2 contains the necessary information about the functional spaces used. In the same section, we introduce the key notions of carrier cones of a functional belonging to $S'^0(\mathbb{R}^d)$ and of a multilinear form defined on $S^0(\mathbb{R}^d)\times\cdots\times S^0(\mathbb{R}^d)$. In Sec. 3, we state Steinmann's regularity conditions and briefly describe the characteristic properties of the retarded products of nonlocal quantum fields. In Sec. 4, we obtain a Paley-Wiener-Schwartz-type theorem for multilinear forms and use this result to establish the correspondence between Steinmann's conditions and asymptotic commutativity. In Sec. 5, we extend the uniqueness theorem obtained in [13] to distributions supported in a properly convex wedge and show that this allows deriving analogues of the CPT and spin-statistics theorems for theories with space-space noncommutativity in exactly the same manner as for nonlocal QFT in [5]. Section 6 contains concluding remarks.

2. Preliminaries

We recall that S^0 is the space of all entire analytic functions satisfying the inequalities

$$|f(x+iy)| \le C_N(1+|x|)^{-N}\, e^{B|y|}, \quad N=0,1,2,\dots,$$

where C_N and B are constants depending on f [14]. To develop a field theory with test functions of this kind, certain spaces related to S^0 and associated with cones in $\mathbb{R}^d$ should be used. If $U\subset\mathbb{R}^d$ is an open cone, then the space $S^0(U)$ is defined as the union of the countably normed spaces $S^{0,b}(U)$, $b>0$, consisting of entire functions with finite norms of the form

$$\|f\|_{U,B,N} = \sup_{z\in\mathbb{C}^d} |f(z)| \prod_{j=1}^{d}(1+|x_j|)^N e^{-Bd(x,U)-B|y|}, \quad B>b,\ N=0,1,2,\dots, \tag{1}$$

where $z=x+iy$ and $d(x,U)=\inf_{\xi\in U}|x-\xi|$ is the distance of x from U. Clearly, $S^0(U)$ is continuously embedded in $S^0(U')$ if $U\supset U'$. Hereafter, we take the norm in $\mathbb{R}^d$ to be $|x|=\sum_j|x_j|$. Norm (1) then has the nice property of multiplicativity, which is used below in treating tensor products. Namely, if $f_1\in S^0(U_1)$ and $f_2\in S^0(U_2)$, then

$$\|f_1\otimes f_2\|_{U_1\times U_2,B,N} = \|f_1\|_{U_1,B,N}\cdot\|f_2\|_{U_2,B,N}. \tag{2}$$

We let S'^0 denote the space of continuous linear functionals on S^0 and call a closed cone $K\subset\mathbb{R}^d$ a carrier of $v\in S'^0(\mathbb{R}^d)$ if the functional v allows a continuous extension to each space $S^0(U)$, where $U\supset K\setminus\{0\}$ (the last inclusion is also written as $U\Supset K$). This amounts to saying that v has a continuous extension to the inductive limit

$$S^0(K) = \varinjlim_{U\Supset K} S^0(U). \tag{3}$$

Let w be a separately continuous multilinear form on $\underbrace{S^0(\mathbb{R}^d)\times\cdots\times S^0(\mathbb{R}^d)}_{n}$ and let $K=K_1\times\cdots\times K_n$, where K_j are closed cones in $\mathbb{R}^d$. It is natural to say that w is carried by K if every K_j is a carrier of all linear functionals on $S^0(\mathbb{R}^d)$ defined by $w(f_1,\dots,f_n)$, where f_i with $i\ne j$ are held fixed. By Schwartz's kernel theorem, for each form w, there exists a unique linear functional $v\in S'^0(\mathbb{R}^{dn})$ such that

$$w(f_1,\dots,f_n) = v(f_1\otimes\cdots\otimes f_n).$$

In the field theory context, this means that the vacuum expectation values of products of fields are identified with certain generalized functions, just as in the usual formalism [1]-[3]. In considering the questions stated in the introduction, we use the following theorems.

Theorem 1. *The space $S^0(\mathbb{R}^d)$ is dense in every space $S^0(U)$, where U is an open cone in $\mathbb{R}^d$.*

Theorem 2. *Every element in ${S'}^0(\mathbb{R}^d)$ has a unique minimal carrier cone.*

Theorem 3. *If a functional $v \in {S'}^0(\mathbb{R}^d)$ is carried by a properly convex cone K, then it has the Laplace transform $\mathbf{u}(\zeta) = (v, e^{iz\zeta})$, which is an analytic function on the tube $T^V = \mathbb{R}^d + iV$, where V is the interior of the dual cone $K^* = \{\eta\colon x\eta \geq 0, \forall x \in K\}$. This function satisfies the condition*

$$|\mathbf{u}(\zeta)| \leq C_{R,V'}\,|\operatorname{Im}\zeta|^{-N_{R,V'}}, \qquad \operatorname{Im}\zeta \in V',\ |\zeta| \leq R, \tag{4}$$

for every $R > 0$ and every $V' \Subset V$. If $\operatorname{Im}\zeta \to 0$ in a fixed V', then $\mathbf{u}(\zeta)$ tends to the Fourier transform of v in the topology of $\mathcal{D}'$. Conversely, every function that is analytic on T^V, where V is an open cone in $\mathbb{R}^d$, and that satisfies (4) is the Fourier-Laplace transform of an element in ${S'}^0(V^*)$.

Theorem 4. *A separately continuous multilinear form on $\underbrace{S^0(\mathbb{R}^d) \times \cdots \times S^0(\mathbb{R}^d)}_{n}$ is carried by a cone $K_1 \times \cdots \times K_n$ if and only if its associated generalized function on $\mathbb{R}^{dn}$ has a continuous extension to the space*

$$S^0(K_1, \ldots, K_n) = \varinjlim_{U_1 \Supset K_1, \ldots, U_n \Supset K_n} S^0(U_1 \times \cdots \times U_n). \tag{5}$$

These theorems are similar to those previously established for another functional class ${S'}^0_\alpha$, but proving them is somewhat more laborious because the topological structure of $S^0(U)$ is more complicated than that of $S^0_\alpha(U)$. Theorem 1 is derived in [13] from a density theorem for $S^0_\alpha(U)$. Theorems 2 and 3 are established in [6]. Definition (5) is an analogue of the definition given in [15] for $S^0_\alpha(K_1, \ldots, K_n)$. Theorem 4 is derivable in the same manner as Theorem 3 in [16] but using Theorem 1 in [17] instead of Lemma 5 in [16]. For the applications under consideration, it is essential that the space $S^0(K_1, \ldots, K_n)$ does not coincide with $S^0(K_1 \times \cdots \times K_n)$ in general. The latter is a subspace of $S^0(K_1, \ldots, K_n)$ but a proper subspace unless $K_j = \{0\}$ and $K_j = \mathbb{R}^d$ for all j (see [15], [16]). If a functional $v \in {S'}^0(K_1 \times \cdots \times K_n)$ extends to $S^0(K_1, \ldots, K_n)$, we say that v is strongly carried by the multiple cone $K_1 \times \cdots \times K_n$.

3. Steinmann's condition

We consider the simplest case of a neutral nonlocal scalar field ϕ. This field is assumed to be an operator-valued generalized function defined on $S^0(\mathbb{R}^4)$ instead of the space $S(\mathbb{R}^4)$ of rapidly decreasing smooth functions, which is commonly used in local QFT [1]-[3]. We assume that ϕ satisfies all the Wightman axioms except the microcausality condition which cannot be formulated in terms of such test functions. Let D denote a common dense domain of the operators $\phi(f)$, $f \in S^0(\mathbb{R}^4)$, in the Hilbert space of states and let $U(\Lambda, a)$ be a unitary representation of the Poincaré group in this space. As in [4], we assume that there exist retarded products $R(x; x_1, \ldots, x_n)$, $n = 1, 2, \ldots$, of fields with the following characteristic properties:

1. the operators

$$R(f) = \int R(x; x_1, \dots, x_n)\, f(x, x_1, \dots, x_n)\, \mathrm{d}x \mathrm{d}x_1 \dots \mathrm{d}x_n, \quad f \in S^0(\mathbb{R}^{4(n+1)}),$$

are defined on the domain D and map it into itself,

2.) $R(f)$ are Hermitian for real f,

3. $R(x) = \phi(x)$,

4. the R products are symmetric in the variables $x_1, \dots, x_n$,

5. the equality

$$R(x; y, x_1, \dots, x_n) - R(y; x, x_1, \dots, x_n) = \\ = -i \sum [R(x; x_{i_1}, \dots, x_{i_\alpha}), R(y; x_{i_{\alpha+1}}, \dots, x_{i_n})]$$

holds, where the sum ranges all partitions of $\{x_1, \dots, x_n\}$ into two subsets,

6. $R(\Lambda x + a; \Lambda x_1 + a, \dots, \Lambda x_n + a) = U(\Lambda, a) R(x; x_1, \dots, x_n) U(\Lambda, a)^{-1}$.

In local field theory, a leading role is played by the causality condition

7. $\operatorname{supp} R(x; x_1, \dots, x_n) \subset \mathbb{K}_n$, where

$$\mathbb{K}_n = \mathbb{R}^4 \times \bar{\mathbb{V}}_- \times \dots \times \bar{\mathbb{V}}_- = \{(x, x_1, \dots, x_n) \colon (x_j - x) \in \bar{\mathbb{V}}_-,\ j = 1, \dots, n\} \quad (6)$$

and $\bar{\mathbb{V}}_- = \{\xi \colon\ \xi^2 \geq 0, \xi^0 \leq 0\}$ is the closed backward light cone.

Steinmann proposed replacing support property 7 with a regularity condition for the retarded products in p-space. The Fourier transform of $\langle \Phi, R(x; x_1, \dots, x_n)\Psi \rangle$, where $\Phi, \Psi \in D$, can be regarded as a distribution in the variables $p_1, \dots, p_n$, and $P = p + p_1 + \dots, +p_n$. Let $\chi(p_1, \dots, p_n)$ denote the distribution resulting from the integration with a test function in P. Then the regularity condition is written as follows.[2]

Condition $\mathcal{R}$. *The function $\chi(p_1, \dots, p_n)$ is analytic on the tubular domain*

$$\mathbb{T}^n_- = \{(p_1, \dots, p_n) \colon\ \operatorname{Im} p_j \in \mathbb{V}_-,\ j = 1, \dots, n\}. \quad (7)$$

For each compact set $Q \subset \mathbb{R}^{4n}$, for every $\gamma > 1$, and for every $R > 0$, there exist a positive constant C and nonnegative integers N_j such that

$$|\chi(p_1, \dots, p_n)| \leq C \prod_{j=1}^{n} |\operatorname{Im} p_{j0}|^{-N_j} \quad (8)$$

for $(p_1, \dots, p_n) \in \mathbb{T}^n_-$, $(\operatorname{Re} p_1, \dots, \operatorname{Re} p_n) \in Q$ and $\gamma |\operatorname{Im} \mathbf{p}_j| \leq |\operatorname{Im} p_{j0}| \leq R$, $j = 1, \dots, n$ (here p_{j0} is the 0-component of the four-vector p_j and $\mathbf{p}_j$ is its spatial part).

[2]In [4], this condition was stated for generalized retarded products, but we restrict our consideration to the simplest case of ordinary retarded products.

By this condition, the analyticity properties of the retarded products of nonlocal fields in p-space are identical to those of the retarded products in the conventional local formalism [3]. Restriction (8) ensures that the analytic function χ has a boundary value belonging to the space $\mathcal{D}'$ of Schwartz distributions. But in contrast to the local theory, no restrictions are imposed on the behavior of χ for large p_j, real or imaginary.

The problem of clarifying the meaning of Condition $\mathcal{R}$ in coordinate space was not posed in [4]. The author of that paper emphasized that this condition, in his opinion, is too complicated and too far from any direct physical interpretation to be accepted as a basic postulate. He suggested regarding it simply as a convenient technical property sufficient for developing a consistent scattering formalism for nonlocal fields, including the proof that asymptotic states and the S-matrix exist. The main result of [4] is that under Condition $\mathcal{R}$ the Lehmann-Symansik-Zimmermann reduction formulas for the S-matrix elements hold in exactly the same form as in local QFT. We now show that this condition when translated into x-space is a natural generalization of support property 7 of the retarded products of local fields and can therefore be considered as an appropriate candidate for the causality condition in nonlocal field theory. We also note that an analogue of the time-ordered T-product can be constructed from the R-products using the usual recursive relations [3] and Condition $\mathcal{R}$ implies that there exists a single analytic function whose boundary values from different domains are the Fourier transforms of the retarded and causal Green's functions $\langle\Psi_0, R(x; x_1, \dots, x_n)\Psi_0\rangle$ and $\langle\Psi_0, T(x; x_1, \dots, x_n)\Psi_0\rangle$, where Ψ_0 is the vacuum state.

4. Relation to asymptotic commutativity

Theorem 4 shows that the space of separately continuous multilinear forms defined on $S^0(\mathbb{R}^d) \times \cdots \times S^0(\mathbb{R}^d)$ and carried by a multiple cone $K_1 \times \cdots \times K_n$ is identified with the dual of space (5). Let $V_1, \dots, V_n$ be open cones in $\mathbb{R}^d$ and let $\mathcal{A}_0(V_1, \dots, V_n)$ denote the space of all functions analytic on the tube $T^V = \mathbb{R}^{dn} + iV$, where $V = V_1 \times \cdots \times V_n$, and satisfying

$$|\mathbf{u}(\zeta)| \le C_{R,V_1',\dots,V_n'} \prod_{j=1}^{n} |\operatorname{Im}\zeta_j|^{-N_{R,V_j'}}, \qquad \operatorname{Im}\zeta_j \in V_j',\ |\zeta_j| \le R,\ j = 1, \dots, n, \tag{9}$$

for each $R > 0$ and for every $V_j' \Subset V_j$. Clearly, $\mathcal{A}_0(V_1, \dots, V_n)$ is an algebra under pointwise multiplication.

Theorem 5. *The Laplace transformation $\mathcal{L}: v \to (v, e^{iz\zeta})$ is an isomorphism of the space $S'^0(V_1^*, \dots, V_n^*)$ onto the algebra $\mathcal{A}_0(V_1, \dots, V_n)$. If $\operatorname{Im}\zeta \to 0$ inside a fixed cone $V_1' \times \cdots \times V_n'$, where $V_j' \Subset V_j$, then the function $(\mathcal{L}v)(\zeta)$ tends to the Fourier transform of v in the topology of $\mathcal{D}'(\mathbb{R}^{dn})$.*

Proof. Because $S'^0(V_1^*, \dots, V_n^*) \subset S'^0(V^*)$, we can apply Theorem 3, whose statement coincides with that of Theorem 5 for $n = 1$ and which, in particular, shows that every functional in $S'^0(V^*)$ has a Laplace transform analytic on T^V. The bound in (9) is stronger then the bound in (4), which holds for an arbitrary element in $S'^0(V^*)$, but can be obtained the same way starting from the estimate

$$|\mathcal{L}v(\zeta)| = |(v, e^{iz\zeta})| \le \|v\|_{U,B,N} \|e^{iz\zeta}\|_{U,B,N},$$

where $U = U_1 \times \cdots \times U_n$, with U_j being any open cones such that $V_j^* \Subset U_j$, B can be taken arbitrarily large, and N generally depends on B and U. Because of (2), we have

$$\|e^{iz\zeta}\|_{U,B,N} = \prod_j \|e^{iz_j\zeta_j}\|_{U_j,B,N},$$

where each factor can be estimated exactly the same way as the norm of an exponential in [6], which yields (9).

We now let u be the boundary value of a function $\mathbf{u}$ with property (9), which exists in $\mathcal{D}'(\mathbb{R}^{dn})$ by Theorem 3.1.15 in [18]. Theorem 3 shows that the cone $V_1^* \times \cdots \times V_n^*$ is a carrier of the multilinear form defined on $S^0(\mathbb{R}^d) \times \cdots \times S^0(\mathbb{R}^d)$ by the inverse Fourier transform of the distribution u. Applying Theorem 4 finishes the proof.

By rewriting the regularity condition for the Green's functions of the nonlocal field theory in terms of Theorem 5, we obtain the following result.

Theorem 6. *Condition $\mathcal{R}$ is equivalent to the requirement that in x-space, the functionals defined on $S^0(\mathbb{R}^{4(n+1)})$ by the matrix elements $\langle \Phi, R(x; x_1, \ldots, x_n)\Psi\rangle$, $\Phi, \Psi \in D$, are strongly carried by cone* (6)*.*

More specifically, Condition $\mathcal{R}$ means that all these functionals allow a continuous extension to any space $S^0(U_\gamma)$, where

$$U_\gamma = \{(x, x_1, \ldots, x_n)\colon |\mathbf{x} - \mathbf{x}_j| < \gamma(x^0 - x_j^0),\ j = 1, \ldots, n\}, \quad \gamma > 1.$$

In particular, when coupled with property 5 in Sec. 3, this condition implies that the matrix elements of the commutator

$$\langle \Phi, [\phi(x), \phi(y)]\Psi\rangle = \langle \Phi, (R(x, y) - R(y, x))\Psi\rangle, \qquad \Phi, \Psi \in D,$$

are strongly carried by the cone $\mathbb{R}^4 \times \bar{\mathbb{V}} = \{(x, y) \in \mathbb{R}^8\colon (x - y)^2 \geq 0\}$. Therefore the theory satisfies the asymptotic commutativity condition [5], which ensures the CPT invariance and the standard spin-statistics relation.

5. An axiomatic formulation of noncommutative QFT

We consider a theory with space-space noncommutativity, namely, with the commutation relation

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{10}$$

where $\theta^{23} = -\theta^{32}$ is a constant noncommutativity parameter and all other matrix entries $\theta^{\mu\nu}$ are zero. Clearly, in the Lorentz group, $O(1,1) \times SO(2)$ is the largest subgroup leaving relation (10) invariant. Its identity component is the product of the group $SO_0(1,1)$ of boosts in the plane (x^0, x^1) and the group $SO(2)$ of rotations in the plane (x^2, x^3). The main idea in [9] is to match the Wightman axioms with this residual symmetry. From such a standpoint, letting $\mathcal{T}_4$ denote the group of space-time translations, we assume the existence of a continuous unitary representation $U(\Lambda, a)$ of the connected group

$$[SO_0(1,1) \times SO(2)] \rtimes \mathcal{T}_4, \tag{11}$$

in the Hilbert space of states and suppose that there is a unique vacuum vector Ψ_0, which is invariant under $U(I,a)=e^{ia^\mu P_\mu}$. The modified spectral condition proposed in [9] implies that the spectrum of the energy-momentum operator P lies in the forward light wedge, i.e.,

$$\mathrm{Spec}\, P \subset \bar{\mathbb{V}}_{c+}\times\mathbb{R}^2=\{p\in\mathbb{R}^4\colon\ p_c^2=p_0^2-p_1^2\geq 0, p_0\geq 0\}. \tag{12}$$

Hereafter, we use the subscript c to denote the restriction of variables to the commutative subspace. Analogously, the microcausality axiom is replaced by the condition that the field commutators (or anticommutators for unobservable fields) $[\phi_\iota(x),\phi_{\iota'}(x')]_{\overset{-}{(+)}}$ vanish outside the closed wedge

$$\bar{\mathbb{V}}_c\times\mathbb{R}^6=\{(x,x')\colon\ (x-x')_c^2\geq 0\}. \tag{13}$$

We restrict our consideration to the case of a complex scalar field ϕ with the transformation law

$$U(\Lambda,a)\phi(f)U^{-1}(\Lambda,a)=\phi(f_{(\Lambda,a)}), \tag{14}$$

where $f_{(\Lambda,a)}(x)=f(\Lambda^{-1}(x-a)$, and we comment on the possibility of generalizing this axiomatic framework using analytic test functions in S^0, as proposed in [12].

We first point out a consequence of conditions (12), (14) that holds independently of a causality formulation. Because of the translation invariance, the n-point vacuum expectation value

$$\mathcal{W}(x_1,\dots,x_n)=\langle\Psi_0,\,\phi^{(*)}(x_1)\dots\phi^{(*)}(x_n)\Psi_0\rangle,$$

where $\phi^{(*)}$ is either ϕ or the Hermitian adjoint field ϕ^*, can be identified with a generalized function $W\in {S'}^0(\mathbb{R}^{4(n-1)})$ in the relative coordinates $\xi_j=x_j-x_{j+1}$, $j=1,\dots,n-1$. Lemma 4 in [5] shows that the Wightman function W is carried by a cone $K\subset\mathbb{R}^{4(n-1)}$ if and only if $\mathcal{W}$ is strongly carried by its inverse image $K\times\mathbb{R}^4$ in $\mathbb{R}^{4n}$ and the membership relation $W\in {S'}^0(K_1,K_2)$ amounts to $\mathcal{W}\in {S'}^0(K_1,K_2,\mathbb{R}^4)$. We use the notation

$$\mathbb{J}^c_{n-1}=\mathbb{V}^{n-1}_{c,R}\cup\mathbb{V}^{n-1}_{c,L}, \tag{15}$$

where $\mathbb{V}_{c,R}=\{(\xi^0,\xi^1)\colon\ (\xi^0)^2-(\xi^1)^2<0,\xi^1>0\}$ and $\mathbb{V}_{c,L}=-\mathbb{V}_{c,R}$. According to [1], two-component open cone (15) consists of the real points of analyticity of the Wightman functions of local field theory in a space-time of two dimensions.

Theorem 7. *Let $\phi(x)$ be a scalar field defined as an operator-valued generalized function on the space $S^0(\mathbb{R}^4)$ and satisfying conditions* (12) *and* (14). *Then the difference*

$$W(\xi_1,\dots,\xi_{n-1})-W(-\xi_1,\dots,-\xi_{n-1}). \tag{16}$$

is strongly carried by the wedge $\complement\mathbb{J}^c_{n-1}\times\mathbb{R}^{2(n-1)}$. In particular, the vacuum expectation value of the commutator

$$\langle\Psi_0,\,[\phi(x),\phi(x')]_-\Psi_0\rangle. \tag{17}$$

is strongly carried by wedge (13).

Proof. The Wightman functions are invariant under the joint inversion of ξ^2 and ξ^3, which is implemented by the $SO(2)$ rotation through 180^0. Therefore, it suffices to prove that the specified wedge is a strong carrier of the difference

$$W(\xi) - W(I_c\xi), \tag{18}$$

where I_c is the inversion of the commutative coordinates ξ^0, ξ^1 and the notation $\xi = (\xi_1, \dots, \xi_{n-1})$ is used for brevity. By Theorem 4, this is equivalent to the statement that the cone $\complement\mathbb{J}^c_{n-1}$ is a carrier of all functionals obtained from (18) by averaging with test functions depending only on ξ^2 and ξ^3. Let W_g be the result of averaging W with such a function $g \in S^0(\mathbb{R}^{2(n-1)})$. By the hypothesis of the theorem, the functional $W_g \in S'^0(\mathbb{R}^{2(n-1)})$ is invariant under the proper orthochronous Lorentz group $SO_0(1,1)$ in $1+1$ dimensions, and its Fourier transform has support in the properly convex cone $\bar{\mathbb{V}}^{(n-1)}_{c+}$. If W is assumed to be a tempered distributions as in [9], then this support property implies that W_g is the boundary value of a function $\mathbf{W}_g(\zeta)$ analytic in the tube $\mathbb{T}^{n-1}_{c-} = \{\zeta = \xi_c + i\eta\colon\ \eta \in \mathbb{V}^{n-1}_{c-}\}$. By the Bargmann-Hall-Wightman theorem [1]–[3], the function $\mathbf{W}_g$ allows an analytic continuation to an extended tube, and after continuation, it is invariant under the proper complex Lorentz group. In the two-dimensional case, this group coincides with the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ of complex numbers, and if we use the light-cone coordinates $\zeta^\pm = (\zeta^0 \pm \zeta^1)/\sqrt{2}$, then we can write its action as

$$\zeta^\pm \to z^{\pm 1}\zeta^\pm, \qquad z \in \mathbb{C}^*. \tag{19}$$

In particular, the complex group contains the inversion I_c, although this transformation does not belong to the identity component of $SO(1,1)$. Cone (15) is obviously contained in the extended tube[3] because $\mathbb{V}^{n-1}_{c,R} = \{\xi_c\colon\ \xi^+ > 0, \xi^- < 0\}$ and the transformation (19) with $z = -i$ carries $\mathbb{V}^{n-1}_{c,R}$ into $i\mathbb{V}^{n-1}_{c-}$. For $\mathbb{V}^{n-1}_{c,L}$, the same role is played by $z = i$. Because g is arbitrary, this immediately implies that in the noncommutative theory [9], which assumes tempered distribution fields, the difference (18) is identically zero in the wedge $\mathbb{J}^c_{n-1} \times \mathbb{R}^{2(n-1)}$.

The ultraviolet behavior of $W_g \in S'^0$ can be regularized by multiplying its Fourier transform by an $SO(1,1)$-invariant function $\omega(P_c^2/\Lambda)$, where $\omega \in C_0^\infty(\mathbb{R})$ and $P_c = \sum_j p_{c,j}$. Theorem 8 in [19] shows that this yields a tempered distribution. For brevity, let G denote the difference (18) averaged with g, and let G_Λ be the result of its regularization. According to the aforesaid, $\operatorname{supp} G_\Lambda$ is contained in the complement of the Jost cone $\mathbb{J}^c_{n-1}$. Therefore, for any Λ, this distribution has a continuous extension $\hat{G}_\Lambda$ to the space $S^0(U)$, where $U \Supset \complement\mathbb{J}^c_{n-1}$, and even to $S^0(\mathbb{U})$, where $\mathbb{U}$ is the interior of the closed cone $\complement\mathbb{J}^c_{n-1}$. Such an extension can be defined by $(\hat{G}_\Lambda, f) = (G_\Lambda, \chi f)$, where χ is an infinitely differentiable function that equals unity in an ϵ-neighborhood of $\mathbb{U}$ and vanishes outside the 2ϵ-neighborhood. Clearly, multiplication by χ maps $S^0(\mathbb{U})$ continuously into the Schwartz space $S(\mathbb{R}^{2(n-1)})$. We can now apply the arguments used to prove the CPT theorem for nonlocal QFT in [13]. Namely, we choose ω such that $\omega(t) = 1$ for $|t| \le 1$ and define a functional $\hat{G}$ on $S^0(\mathbb{U})$ by

$$(\hat{G}, f) = (\hat{G}_\Lambda, f), \quad \text{where} \quad \Lambda = 4\sqrt{2n^3}eb \quad \text{if} \quad f \in S^{0,b}(\mathbb{U}). \tag{20}$$

[3]We note that the description [1] of the real points of analyticity for any dimension has been obtained just by reducing to this simple two-dimensional case.

Theorem 1 ensures that $\hat{G}$ is well defined. According to its detailed formulation given in [13], the space $S^{0,b'}(\mathbb{R}^d)$ is dense in $S^{0,b}(U)$ in the topology of $S^{0,b'}(U)$ for each open cone $U \subset \mathbb{R}^d$ and for every pair of positive numbers b and b' such that $b' > 2deb$. Hence, there exists a sequence $f_\nu \in S^{0,4neb}(\mathbb{R}^{2(n-1)})$ such that $f_\nu \to f$ in $S^{0,4neb}(\mathbb{U})$. By the Paley-Wiener-Schwartz theorem (Theorem 7.3.1 in [18]), the Fourier transforms of the functions f_ν have support in the hypercube $\max_j(|p_j^0|, |p_j^1|) \le 4neb$, where $|P_c^2| \le 32n^3e^2b^2$ and where $\omega(P_c^2/\Lambda^2) \equiv 1$ with our choice of Λ. Therefore $(\hat{G}_\Lambda, f_\nu) = (\hat{G}_{\Lambda_1}, f_\nu) = (G, f_\nu)$ if $b_1 > b$, and hence $(\hat{G}_\Lambda, f) = (\hat{G}_{\Lambda_1}, f)$. This consideration also shows that $\hat{G}$ is an extension of G and is continuous in the topology of $S^0(\mathbb{U})$, which finishes the proof.

Theorem 7 shows that if we assume weak form (12) of the spectral condition and use the test function space S^0, then an adequate modification of the microcausality axiom is the requirement that wedge (13) be a strong carrier of the matrix elements of the field commutators or anticommutators $\langle\Phi,\, [\phi_\iota(x), \phi_{\iota'}(x')]_\mp\Psi\rangle$ for any Φ and Ψ in a common dense invariant domain D of fields in the Hilbert space of states.

We now show that the uniqueness theorem established in [13] for the distributions supported by a properly convex cone allows a direct generalization to the case of a wedge-shaped support.

Theorem 8. *Let $u \in \mathcal{D}'(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ be a nontrivial distribution whose support is contained in the wedge $V \times \mathbb{R}^{d_2}$, where V is a properly convex cone. Then every strong carrier cone of its Fourier transform $\tilde{u} \in S'^0(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ has the form $\mathbb{R}^{d_1} \times K_2$, where K_2 is a closed cone in $\mathbb{R}^{d_1}$.*

Proof. We assume that this is not the case and $\tilde{u} \in S'^0(K_1, K_2)$, where $K_1 \ne \mathbb{R}^{d_1}$. Then for each $g \in S^0(\mathbb{R}^{d_2})$, the functional $\tilde{u}_g$ defined by $(\tilde{u}_g, f) = (\tilde{u}, f \otimes g)$ is carried by the cone K_1, as can be readily seen from (2). The distribution on $\mathbb{R}^{d_1}$ whose Fourier transform is $\tilde{u}_g$ has support in V. Hence this distribution is trivial by Theorem 4 in [13]. Because g is arbitrary and $\mathcal{D}(\mathbb{R}^{d_1}) \otimes \mathcal{D}(\mathbb{R}^{d_2})$ is dense in $\mathcal{D}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$, we infer that $u \equiv 0$ and obtain a contradiction.

Theorem 8 provides a way to extend some of the results in [5], [13] on the spin-statistics relation and the CPT symmetry to this version of noncommutative field theory. Let

$$W_{\phi\phi^*}(x - x') = \langle\Psi_0,\, \phi(x), \phi^*(x')\Psi_0\rangle, \qquad W_{\phi^*\phi}(x - x') = \langle\Psi_0,\, \phi^*(x), \phi(x')\Psi_0\rangle.$$

As a simplest example, we show that the wedge (13) cannot be a strong carrier of the anticommutator $[\phi(x), \phi^*(x')]_+$, i.e., the anomalous commutation relation is forbidden for the scalar field. Indeed, the functional $W_{\phi\phi^*} + W_{\phi^*\phi}$ is otherwise strongly carried by the wedge $\bar{\mathbb{V}}_c \times \mathbb{R}^2$ by Theorem 7. From (12), it follows that the momentum-space support of this sum lies in the wedge $\bar{\mathbb{V}}_{c+} \times \mathbb{R}^2$. Therefore, this functional is zero by Theorem 8. Averaging with a test function of the form $\bar{f}(x)f(x')$, we obtain $\|\phi^*(f)\Psi_0\|^2 + \|\phi(f)\Psi_0\|^2 = 0$ and hence

$$\phi(f)\Psi_0 = \phi^*(f)\Psi_0 = 0 \qquad \text{for all} \quad f \in S^0(\mathbb{R}^4).$$

As in the usual local theory [2], [3], this implies that the field ϕ vanishes. The arguments are analogous to those in the proof of Theorem 13 in [5] and again use the above generalization of the uniqueness theorem.

It is well known that the existence of CPT symmetry in the theory of a scalar field is equivalent to the validity of the relations

$$W(\xi_1,\dots,\xi_{n-1}) = \breve{W}(\xi_{n-1},\dots,\xi_1), \tag{21}$$

where $\breve{W}$ is defined by the same set of field operators as W but taken in the inverse order. The functional

$$W(\xi_1,\dots,\xi_{n-1}) - \breve{W}(\xi_{n-1},\dots,\xi_1) \tag{22}$$

can be written as

$$[W(\xi_1,\dots,\xi_{n-1}) - \breve{W}(-\xi_{n-1},\dots,-\xi_1)] + [\breve{W}(-\xi_{n-1},\dots,-\xi_1) - \breve{W}(\xi_{n-1},\dots,\xi_1)]. \tag{23}$$

According to Theorem 7, the expression in the second square brackets in (23) is strongly carried by the wedge $\complement\mathbb{J}^c_{n-1}\times\mathbb{R}^{2(n-1)}$. If the modified causality condition is satisfied with the standard commutation relation, then the functional in the first square brackets in (23) is also strongly carried by this wedge. In fact, this functional expresses the difference between two vacuum expectation values, one obtained from the other by permuting the fields. This difference is representable as a sum of terms of the form $\langle\Psi_0,\dots[\phi^{(*)}(x_i),\phi^{(*)}(x_j)]_-\dots\Psi_0\rangle$, where the dots stand for field operators. Such a term is strongly carried by the wedge $\bar{\mathbb{V}}_c\times\mathbb{R}^{4n-2} = \{x\colon\ (x_i-x_j)^2_c\ge 0\}$, whose image $\{\xi\colon\ (\xi_i+\dots+\xi_{j-1})^2_c\ge 0\}$ in the relative coordinate space is contained in $\complement\mathbb{J}^c_{n-1}\times\mathbb{R}^{2(n-1)}$. It follows from (12) that the momentum-space support of functional (22) lies in the wedge $\bar{\mathbb{V}}^{n-1}_{c+}\times\mathbb{R}^{2(n-1)}$. Applying Theorem 8 again, we conclude that the noncommutative scalar field theory with the functional domain $S^0(\mathbb{R}^4)$, satisfying spectral condition (12) and the above-stated modification of microcausality axiom, has CPT as a symmetry. Moreover, the presence of CPT symmetry is tantamount to the condition that all possible functionals in the first square brackets in (23) are strongly carried by the wedges $\complement\mathbb{J}^c_{n-1}\times\mathbb{R}^{2(n-1)}$, $n=2,3,\dots$, which is a generalization of weak local commutativity.

6. Conclusion

The established equivalence of asymptotic commutativity to certain regularity properties of the retarded Green's functions in momentum space once again shows that the formulation of nonlocal QFT using highly singular generalized functions is quite rich in physical content because it has a self-consistent interpretation in terms of particle scattering together with the CPT symmetry and the standard spin-statistics relation. We note that the distinction between carrier cones and strong carrier cones is typical not only of the functional class S'^0, but also of the generalized functions defined on any one of the Gelfand-Shilov spaces S^β and S^β_α with the superscript $\beta\le 1$. This is particularly true in regard to the Fourier hyperfunctions, which compose the dual space of S^1_1 and provide the most general framework for constructing a local field theory. We have commented on only one attempt to extend the axiomatic approach to noncommutative field theories. It will be instructive to consider the possibility of using analytic test functions in other formulations, particularly in the construction of Wightman functions using the Moyal $*$-product instead of the ordinary product of fields and with the implementation of twisted Poincaré symmetry, as proposed in [20], but this is beyond the scope of the present paper.

Acknowledgments. This paper was supported in part by the the Russian Foundation for Basic Research (Grant No. 05-01-01049) and the Program for Supporting Leading Scientific Schools (Grant No. LSS-1578.2003.2).

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